The Psi function and its zeros on the complex plane

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The Psi function

I defined the **Psi function** as the sum of reciprocals of factorials to the s power:

$$\psi(s) := \sum_{n=1}^{\infty} \frac{1}{(n!)^s} \qquad s \in \mathbb{C} \tag{1}$$

The zeros of the Psi function on the complex plane

I found interesting studying **the zeros of the Psi function** from a computational point of view with MATLAB/Octave and Mathematica.

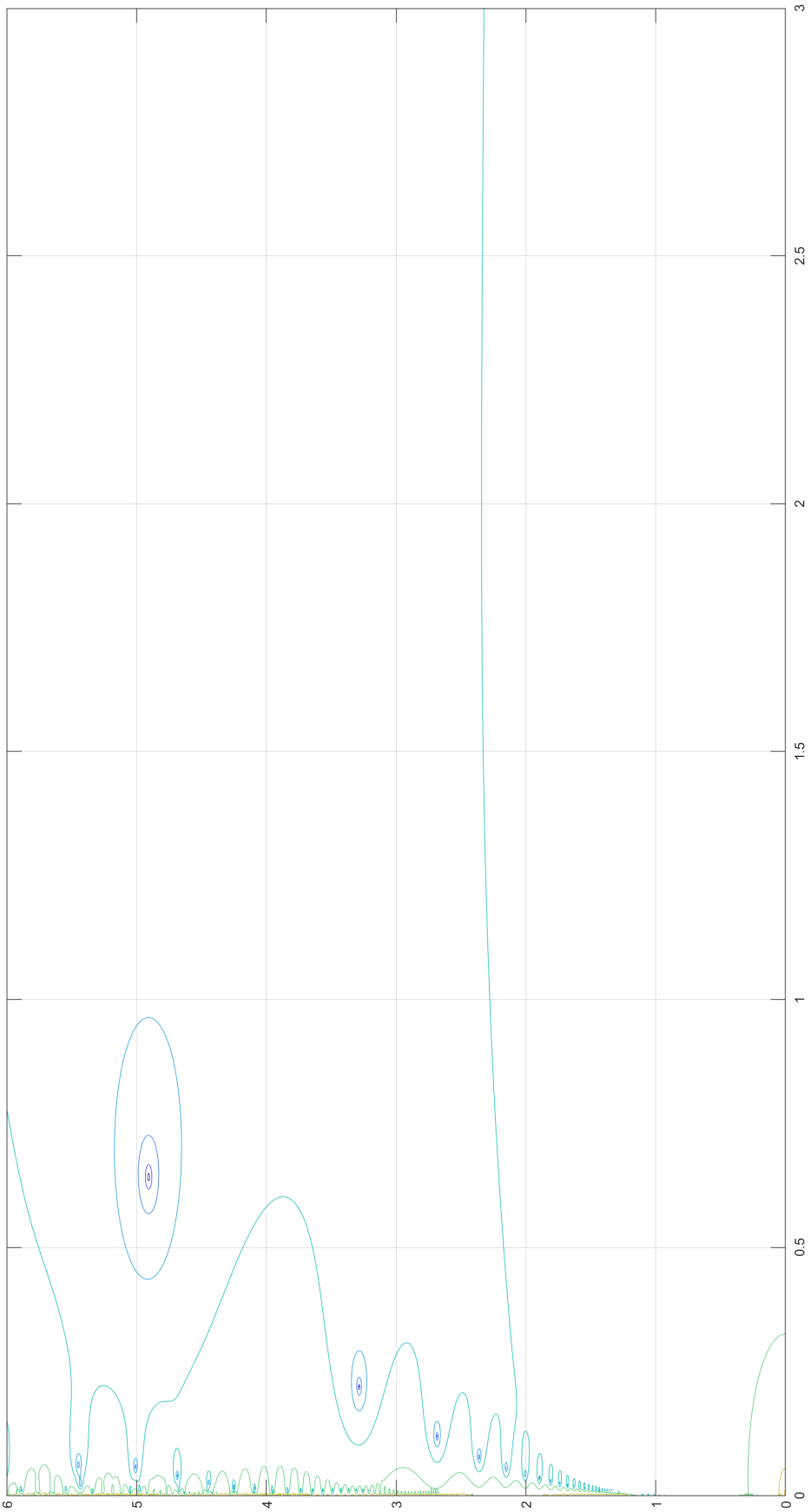
I formulate two conjectures:
Conjecture 1: the non-trivial zeros of the Psi function are all in the **critical strip** $0 < \mathrm{Re}(s) \leqslant 1$.
Conjecture 2: the non-trivial zeros of the Psi function are all in the strip $0 < \mathrm{Re}(s) \leqslant 0.73$.

A proposal for a new mathematical constant

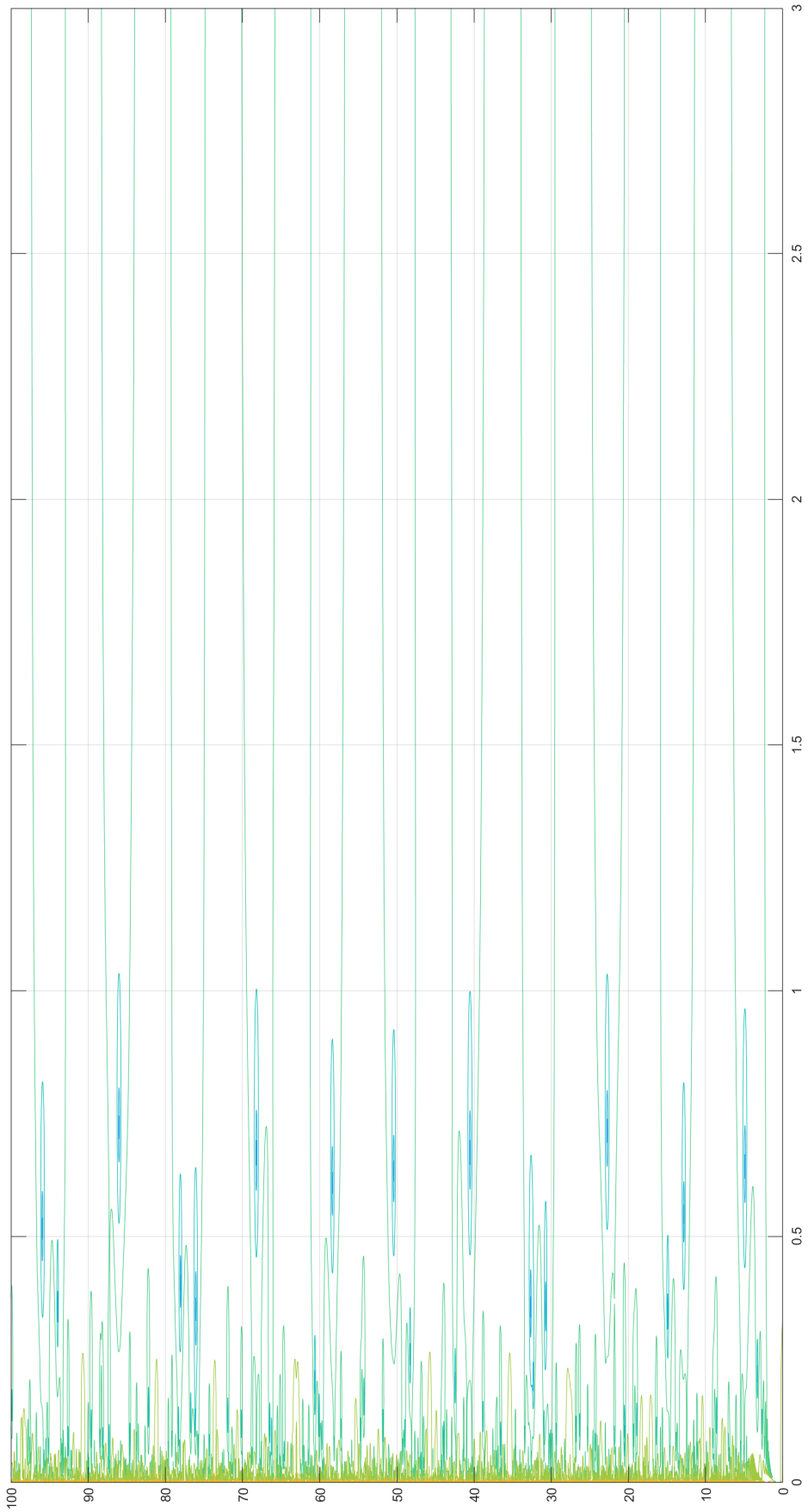
Below is a sequence of graphs of the levels (contour) of the function log10(abs(psi(s))) for s belonging to the complex plane.

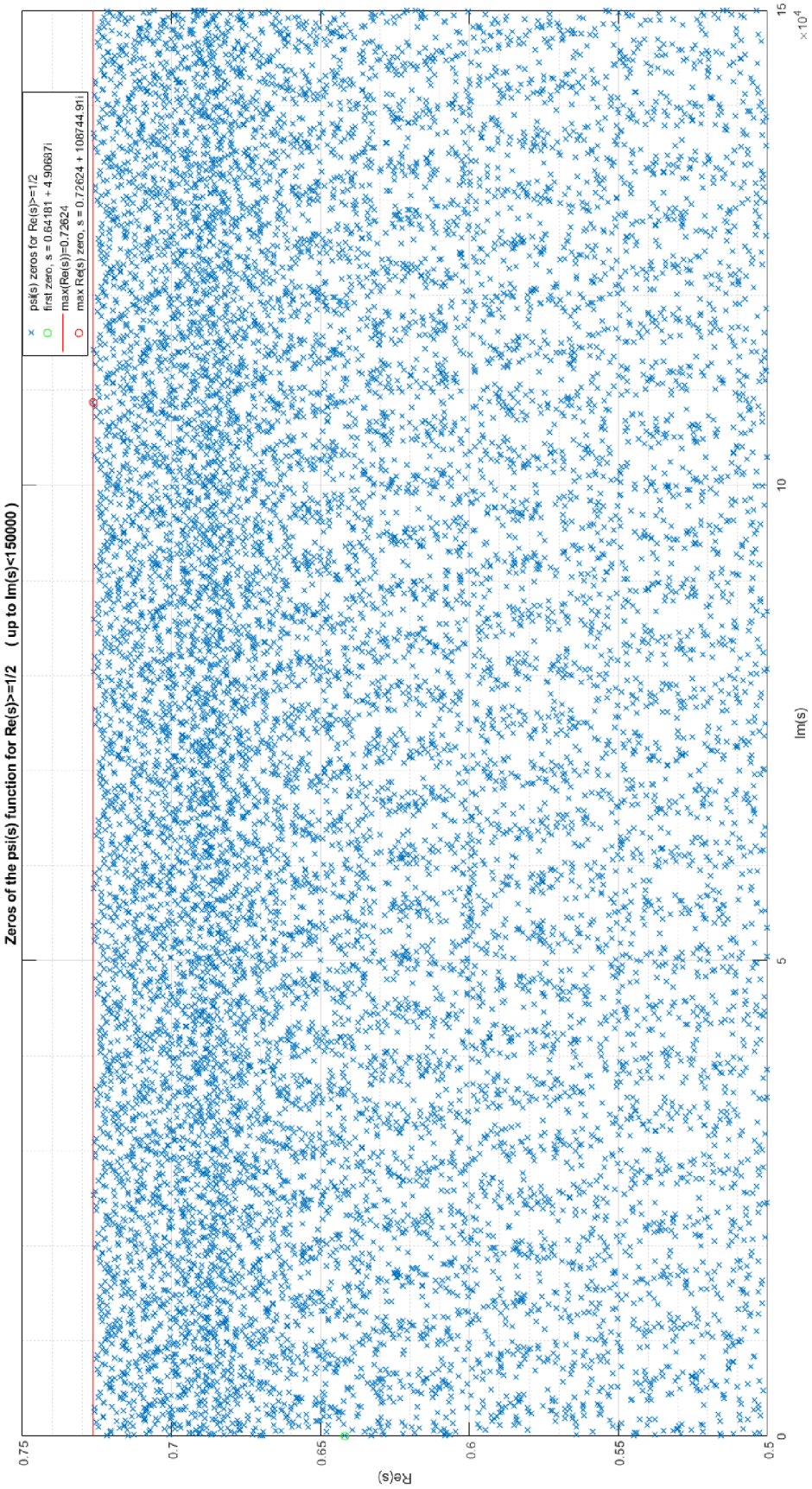
In the first graph, the **first zero in the semi-plane Re(s)$\geqslant\frac{1}{2}$** is in evidence.
His position is approximately $\boldsymbol{s_1 \simeq 0.6418158643 + 4.9068764351i}$.

Im(s_1) ≃ 4.9068764351 seems to be **a new mathematical constant**.
A search in the OEIS Foundation database https://oeis.org/ is negative.
Also unsuccessful was a search for a composed constant (see at the end).


6
5
4
3
2
1
0
0
0.5
1
1.5
2
2.5
3

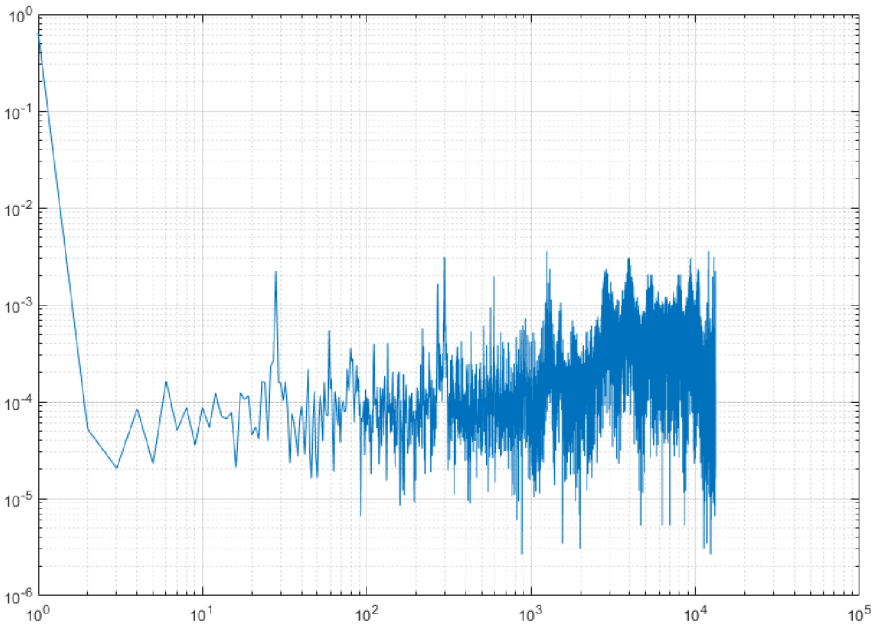
20
18
16
14
12
10
8
6
4
2
0
0
0.5
1
1.5
2
2.5
3


100
90
80
70
60
50
40
30
20
10
0
0
0.5
1
1.5
2
2.5
3

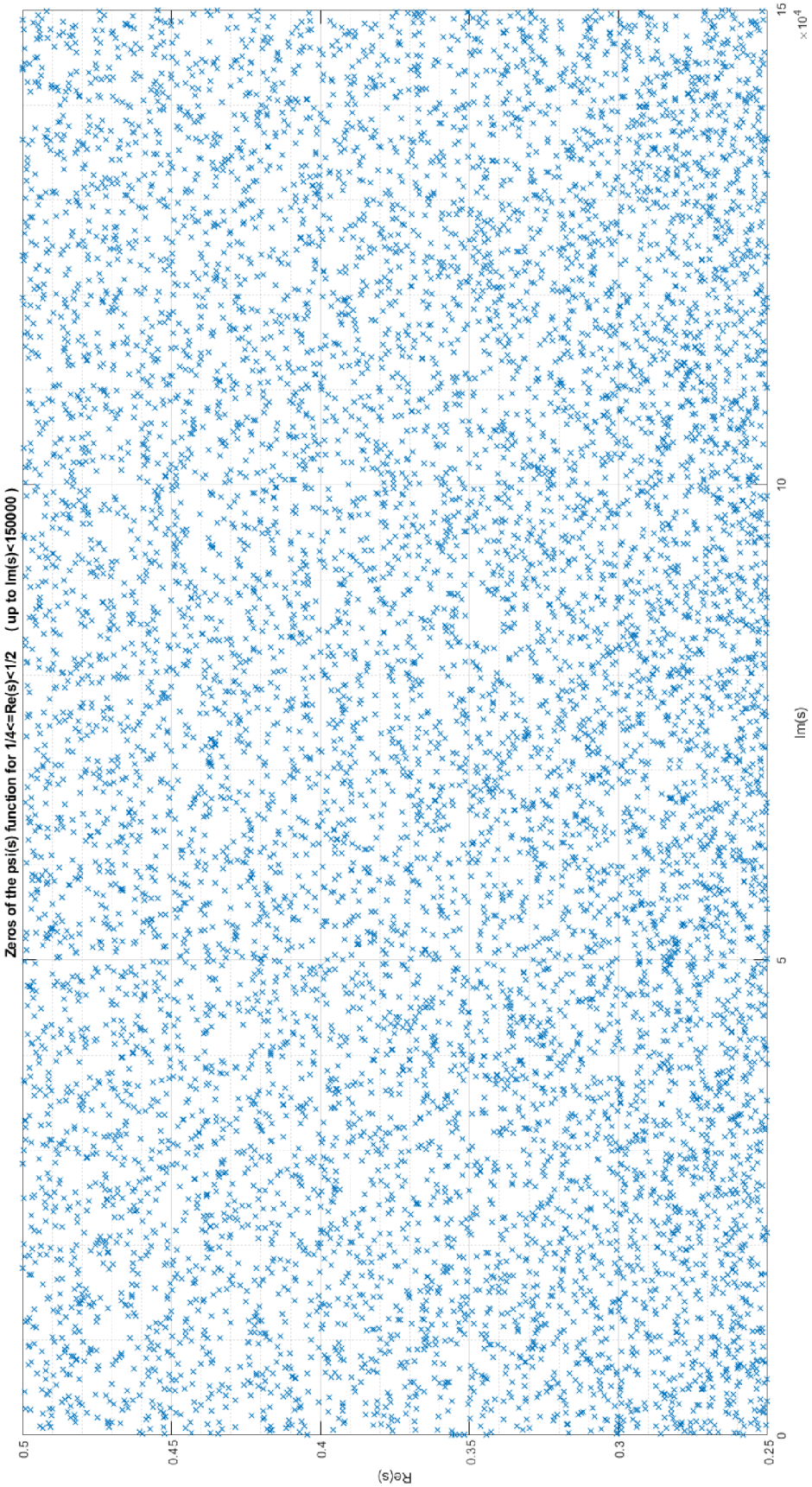

Zeros of the psi(s) function for Re(s)>=1/2 (up to Im(s)<150000)
psi(s) zeros for Re(s)>=1/2
first zero, s = 0.64181 + 4.90687i
max(Re(s))=0.72624
max Re(s) zero, s = 0.72624 + 106744.91i
Re(s)
Im(s)
0.75
0.7
0.65
0.6
0.55
0.5
0
5
10
15
×10^4

Seeing the previous graph of the first 13283 zeros ($0 \leqslant \mathrm{Im}(s) \leqslant 150000$) for $\mathrm{Re}(s) \geqslant \frac{1}{2}$, it seems that some kind of periodic patten is present in the positions of the zeros. An FFT analysis of the coordinates of the zeros reveals something similar.
Furthermore, both conjectures 1 and 2 seem true.

FFT analysis of the zeros real part (for $\mathrm{Re}(s) \geqslant 1/2$)



In the successive graph there are the first 9486 zeros ($0 \leqslant \mathrm{Im}(s) \leqslant 150000$) for the strip $\frac{1}{4} \leqslant \mathrm{Re}(s) < \frac{1}{2}$.

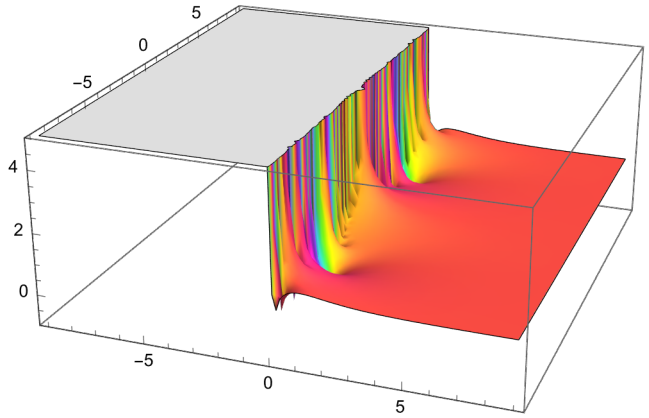

Zeros of the psi(s) function for 1/4<=Re(s)<1/2 (up to Im(s)<150000)
Re(s)
0.5
0.45
0.4
0.35
0.3
0.25
Im(s)
0
5
10
15
×10^4

In[•]:=
```
(*   Approximate Psi(s) function   *)

Psi[s_, nmax_] := ParallelSum[1 / n!^s, {n, 1, nmax}]
```

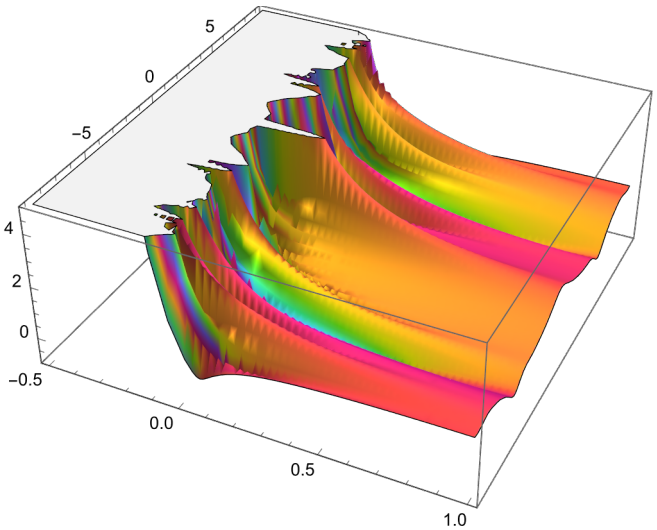
```
ComplexPlot3D[Psi[s, 10], {s, -9 - 9 I, 9 + 9 I}, PlotRange → {-1, 5}]
```

Out[•]=




```
ComplexPlot3D[Psi[s, 10], {s, -1 / 2 - 9 I, 1 + 9 I}, PlotRange → {-1, 5}]
```

Out[•]=



```
(*   Examples of real values   *)

N[Psi[1000, 200], 100]    (*   limit   Psi(s) --Re(s)→ ∞--> 1   *)
```

Out[∘]=

```
1.0000000000000000000000000000000000000000000000000000000000000000000000000000000000
 000000000000000000
```

```
N[Psi[1, 200], 100]     (*    Psi(1) = e – 1   *)
```

Out[∘]=

```
1.7182818284590452353602874713526624977572470936999595749669676277240766303535475945
 71382178525166427
```

In[∘]:=

```
N[Psi[1 / 2, 200], 100]
```

Out[∘]=

```
2.4695063145210475624756367446601502576897561839949691996779257019718354880034088340
 38662508412007135
```

In[∘]:=

```
N[Psi[1 / 4, 300], 100]
```

Out[∘]=

```
3.7063221734793910732444276016472558137744061380342803382068041093450018356719647449
 61441193290389664
```

In[∘]:=

```
N[Psi[5 / 1000, 20000], 100]
```

Out[∘]=

```
58.356775821716875223061509326910407324145736718139894476755786989329630308133281342
 27727344209568700
```

In[∘]:=

```
N[Psi[1 / 1000, 20000], 100]
```

Out[∘]=

```
212.89343444045988644216289367877454278023323209715002451741002064447015144892838658
 14643740128295603
```

```
(*   The s1 zero   *)
```

In[•]:=
```
FindRoot[{Re[Psi[x + y * I, 1000]], Im[Psi[x + y * I, 1000]]},
 {{x, 1 / 2}, {y, 5}}, WorkingPrecision → 50]
```

Out[•]=
```
{x → 0.64181586433314113248112139394951847372318338379012,
 y → 4.9068764351428513475351082583558535315328564648993}
```

In[•]:=
```
FindRoot[{Re[Psi[x + y * I, 1000]], Im[Psi[x + y * I, 1000]]},
 {{x, 1 / 2}, {y, 5}}, WorkingPrecision → 150]
```

Out[•]=
```
{x →
  0.641815864333141132481121393949518473723183383790117174605186222762580517064969791
   3627621271450581796335347122086694672837145467988486316253164421367 67, y →
  4.9068764351428513475351082583558535315328564648993376352028895248700809684916040 6
   0111065196944920181067326293799784810249860650082932557998914248779 82}
```

In[•]:=
```
FindRoot[{Re[Psi[x + y * I, 1000]], Im[Psi[x + y * I, 1000]]},
 {{x, 1 / 2}, {y, 5}}, WorkingPrecision → 200]
```

Out[•]=
```
{x →
  0.641815864333141132481121393949518473723183383790117174605186222762580517064969791
   3627621271450581796335347122086694672837145467988486316253164421367 6681386342214
   0144736772863285465234270809262154826 59, y →
  4.9068764351428513475351082583558535315328564648993376352028895248700809684916040 6
   0111065196944920181067326293799784810249860650082932557998914248779 81697719647867
   8924328658509477678673596964581898501 1}
```

```
(*   Other examples of zeros   *)
```

In[•]:=
```
FindRoot[{Re[Psi[x + y * I, 2000]], Im[Psi[x + y * I, 2000]]},
 {{x, 1336 / 1000000}, {y, 1 + 4705 / 1000000}}, WorkingPrecision → 50]
```

Out[•]=
```
{x → 0.0013361234465891396568602622971086091779190211534074,
 y → 1.0047047214630754990772104173473400204889209870526}
```

In[•]:=
```
FindRoot[{Re[Psi[x + y * I, 20000]], Im[Psi[x + y * I, 20000]]},
 {{x, 1336 / 1000000}, {y, 1 + 4705 / 1000000}}, WorkingPrecision → 50]
```

Out[•]=
```
{x → 0.0013361234423282565356094057650499049578042780328027,
 y → 1.0047047214631007857421115748599516575637423334916}
```

In[•]:=
```
FindRoot[{Re[Psi[x + y * I, 1000]], Im[Psi[x + y * I, 1000]]},
 {{x, 670 / 1000}, {y, 40 + 516 / 1000}}, WorkingPrecision → 50]
```

Out[•]=
```
{x → 0.67022547274446564754942561921853627239573382162093,
 y → 40.516540228521088324200375633146033781651332005560}
```

In[•]:=
```
FindRoot[{Re[Psi[x + y * I, 1000]], Im[Psi[x + y * I, 1000]]},
 {{x, 721 / 1000}, {y, 86 + 2 / 1000}}, WorkingPrecision → 50]
```

Out[•]=
```
{x → 0.72159477459801208832092300307574122028151736384671,
 y → 86.002031717653971712641931481424395869298288479074}
```

In[•]:=
```
FindRoot[{Re[Psi[x + y * I, 1000]], Im[Psi[x + y * I, 1000]]},
 {{x, 200 / 1000}, {y, 512 + 787 / 1000}}, WorkingPrecision → 50]
```

Out[•]=
```
{x → 0.19977206033939305741111784335367081276729571786498,
 y → 512.78788254499022664698311303017535684680849482170}
```

In[•]:=
```
FindRoot[{Re[Psi[x + y * I, 1000]], Im[Psi[x + y * I, 1000]]},
 {{x, 611 / 1000}, {y, 1019 + 180 / 1000}}, WorkingPrecision → 50]
```

Out[•]=
```
{x → 0.61144000424789254536789403748651182040962005969469,
 y → 1019.1796100394148909618609953176907836874492265833}
```

```
0.6919053669 + 149998.1372808481 i
```

```
In[•]:= FindRoot[{Re[Psi[x + y * I, 1000]], Im[Psi[x + y * I, 1000]]},
  {{x, 692 / 1000}, {y, 149998 + 137 / 1000}}, WorkingPrecision → 50]
```

```
Out[•]=
{x → 0.69190536689323821234606604538539980100012880561469,
 y → 149998.13728084803537276334254075480663772664388012}
```

```
0.726240868445895 + 108744.911314453 i
(*   The zero with maximum Re(s)    ( Im(s)<150000 )  *)
```

```
In[•]:= FindRoot[{Re[Psi[x + y * I, 1000]], Im[Psi[x + y * I, 1000]]},
  {{x, 726 / 1000}, {y, 108744 + 911 / 1000}}, WorkingPrecision → 50]
```

```
Out[•]=
{x → 0.72624086844586790634634422045970079610134134053575,
 y → 108744.91131445345894385570366150252207947684867118}
```

```
(*   Factorial   *)
Table[{n, n!}, {n, 1, 40}] // Column
```

Out[•]=

```
{1, 1}
{2, 2}
{3, 6}
{4, 24}
{5, 120}
{6, 720}
{7, 5040}
{8, 40 320}
{9, 362 880}
{10, 3 628 800}
{11, 39 916 800}
{12, 479 001 600}
{13, 6 227 020 800}
{14, 87 178 291 200}
{15, 1 307 674 368 000}
{16, 20 922 789 888 000}
{17, 355 687 428 096 000}
{18, 6 402 373 705 728 000}
{19, 121 645 100 408 832 000}
{20, 2 432 902 008 176 640 000}
{21, 51 090 942 171 709 440 000}
{22, 1 124 000 727 777 607 680 000}
{23, 25 852 016 738 884 976 640 000}
{24, 620 448 401 733 239 439 360 000}
{25, 15 511 210 043 330 985 984 000 000}
{26, 403 291 461 126 605 635 584 000 000}
{27, 10 888 869 450 418 352 160 768 000 000}
{28, 304 888 344 611 713 860 501 504 000 000}
{29, 8 841 761 993 739 701 954 543 616 000 000}
{30, 265 252 859 812 191 058 636 308 480 000 000}
{31, 8 222 838 654 177 922 817 725 562 880 000 000}
{32, 263 130 836 933 693 530 167 218 012 160 000 000}
{33, 8 683 317 618 811 886 495 518 194 401 280 000 000}
{34, 295 232 799 039 604 140 847 618 609 643 520 000 000}
{35, 10 333 147 966 386 144 929 666 651 337 523 200 000 000}
{36, 371 993 326 789 901 217 467 999 448 150 835 200 000 000}
{37, 13 763 753 091 226 345 046 315 979 581 580 902 400 000 000}
{38, 523 022 617 466 601 111 760 007 224 100 074 291 200 000 000}
{39, 20 397 882 081 197 443 358 640 281 739 902 897 356 800 000 000}
{40, 815 915 283 247 897 734 345 611 269 596 115 894 272 000 000 000}
```

```
(*   Some examples of contributions to the summation   *)
```

In[∘]:=
```
N[1 / (200!)^(1 / 2)]
```

Out[∘]=

3.56087×10^{-188}

In[∘]:=
```
N[1 / (200!)^(1 / 4)]
```

Out[∘]=

1.88703×10^{-94}

In[∘]:=
```
N[1 / (200!)^(1 / 1000)]
```

Out[∘]=

0.421797

In[∘]:=
```
N[1 / (2000!)^(1 / 1000)]
```

Out[∘]=

1.83857×10^{-6}

In[∘]:=
```
N[1 / (20000!)^(1 / 1000)]
```

Out[∘]=

4.59981×10^{-78}

```
(*   Im(s1) :  Search for a composed constant   *)

zero = 4.906876435142851347535108258355853 5315
lista = {Pi, Pi^3, Sqrt[Pi], E, E^2, E^3, Sqrt[E], Zeta[2], Zeta[3], Zeta[4], Zeta[5],
  1 / 2, 1 / 3, 1 / 4, 1 / 5, 1 / 7, 1 / 9, 1 / 11, 1 / 13, 2, 3, 4, 5, 7, 8, 9, 10, 11, 13, Sqrt[2],
  Sqrt[3], Sqrt[5], Sqrt[6], Sqrt[7], Sqrt[10], Erf[1], GoldenRatio, GoldenAngle, Sin[1], EulerGamma,
  -StieltjesGamma[1], Catalan, Glaisher, Khinchin, ChampernowneNumber[10]}
lista2 = (DeleteDuplicates[Subsets[#1, 6]] &) [lista]
lista3 = Exp[Total[Log[lista2], {2}]]
N[lista3, 10]
(*  N[Abs[lista3-zero],6]  *)
delta = lista3 - zero;
If[Min[Abs[delta]] < 0.0000001, 1000, 0]
If[Min[Abs[delta + 1]] < 0.0000001, 1000, 0]
If[Min[Abs[delta - 1]] < 0.0000001, 1000, 0]
If[Min[Abs[delta + 1 / 2]] < 0.0000001, 1000, 0]
If[Min[Abs[delta - 1 / 2]] < 0.0000001, 1000, 0]
```

Out[•]=

4.906876435142851347535108258355853 5315

Out[•]=

$\{\pi, \pi^3, \sqrt{\pi}, e, e^2, e^3, \sqrt{e}, \frac{\pi^2}{6}, \text{Zeta}[3], \frac{\pi^4}{90}, \text{Zeta}[5], \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \frac{1}{11}, \frac{1}{13}, 2, 3, 4, 5, 7, 8, 9, 10, 11, 13, \sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{6}, \sqrt{7}, \sqrt{10}, \text{Erf}[1], \text{GoldenRatio}, \text{GoldenAngle}, \text{Sin}[1], \text{EulerGamma}, -\text{StieltjesGamma}[1], \text{Catalan}, \text{Glaisher}, \text{Khinchin}, \text{ChampernowneNumber}[10]\}$

Out[•]=

```
{{}, {π}, {π^3}, {Sqrt[π]}, {e}, ... 9531031 ...,
 {Sin[1], EulerGamma, -StieltjesGamma[1], Glaisher, Khinchin, ChampernowneNumber[10]},
 {Sin[1], EulerGamma, Catalan, Glaisher, Khinchin, ChampernowneNumber[10]},
 {Sin[1], -StieltjesGamma[1], Catalan, Glaisher, Khinchin, ChampernowneNumber[10]},
 {EulerGamma, -StieltjesGamma[1], Catalan, Glaisher, Khinchin, ChampernowneNumber[10]}}
```

Full expression not available (original memory size: 4.2 GB)

Out[•]=

```
{1, π, π^3, Sqrt[π], e, ... 9531031 ...,
 -EulerGamma Glaisher Khinchin ChampernowneNumber[10] Sin[1] StieltjesGamma[1],
 Catalan EulerGamma Glaisher Khinchin ChampernowneNumber[10] Sin[1],
 -Catalan Glaisher Khinchin ChampernowneNumber[10] Sin[1] StieltjesGamma[1],
 -Catalan EulerGamma Glaisher Khinchin ChampernowneNumber[10] StieltjesGamma[1]}
```

Full expression not available (original memory size: 3 GB)

Out[•]=

```
{1.000000000, 3.141592654, 31.00627668, 1.772453851, 2.718281828,
 ... 9531031 ..., 0.01503724305, 0.1891565932, 0.02386213352, 0.01636847558}
```

Full expression not available (original memory size: 0.9 GB)

Out[•]=

0

Out[•]=

0

Out[•]=

0

Out[•]=

0

Out[•]=

0

```
(*   Im(s1) :  Continued fraction analysis   *)

FindRoot[{Re[Psi[x + y * I, 2000]], Im[Psi[x + y * I, 2000]]},
 {{x, 642 / 1000}, {y, 4 + 907 / 1000}}, WorkingPrecision → 1300]
```

Out[•]=

```
{x →
  0.641815864333141132481121393949518473723183383790117174605186222762580517064969 79
   1362762127145058179633534712208669467283714546798848631625316442136766813863422 14
   0144736772863285465234270809262154826594798462499167724922899818816022372831185 20
   2244417333055938758585385009631615835727557064789061796210218635823932399542713 00
   5035020428566874351145213576501840291383578362795976410183239770792832980404586 96
   7312339528521129358353919704260945631364196979304647405392246047746221392704103 10
   8833007736708878687232557146057583614726268598535166340289104925111804264340286 87
   4690006192715473763219212671830536220383560289681640509438660849049748902742923 74
   4478049605265257678100022870495747487263280862917898516955252854621624940265075 74
   4094287931970814747843750427566189358391783641205919945898239215477514139339496 64
   9281699054692870019470766764937806602782627790085117975838802936653254255912409 43
   4401972379709258986484819042121491842716543229014555682410211095786302412686646 21
   4780717062024340142647497025049896164155059636293669416528058572846052742844744 55
   7074096343552177145803703551685091423927336876796888540056645530584716385470303 21
   4637978834244774412088608651956792496336258511265779672039153886517761182555550 97
   5221652296132714275914231641337837083526849052776766012482740869472783550498650 55
   38354, y →
  4.906876435142851347535108258355853531532856464899337635202889524870080968491604 06
   0111065196944920181067326293799784810249860650082932557998914248779816977196478 67
   8924328658509477678673596964581898501130662907306611633829595637112836362465545 38
   6259311006169595007178214401008312972843537525638997252224867324240802468810244 36
   4768226989775452878521175884676279833784979282372041334392007602201764553324130 95
   9276186700171069751255379690909012820599699469693129362082577073655560060041792 29
   4655301531232642872112464958250438382077289092265651034942633975624239680074208 95
   8726094753410805606190690665278381412759451065302650085037363487610006630729810 81
   3165122126812754786330344118466691356717904448534713726380603029604210004054846 83
   8771425836970786349664824147663633944854393850500487390514241108209352915338035 87
   6396883306949926813216340923786343369423300729994233079933817906091951310592780 99
   4262238542959902040323474324465545743357089799988821995107157833270026815488792 43
   6054240133699741905847329101368437182707601454192408809933135631089196893186880 74
   6998141384100711943436137965375195496180721519766987388584353277323481773656338 53
   5496792218282533660207571536553416179289067935583847818396896669365922120992460 55
   0450830106546240826746256595765893238318595226963854794396575502704864042806084 34
   4431}
```

In[ ]:=
```
zero = %[[2]][[2]]
```

Out[ ]=
```
4.906876435142851347535108258355853531532856464899337635202889524870080968491604060 1
 1106519694492018106732629379978481024986065008293255799891424877981697719647867892 4
 3286585094776786735969645818985011306629073066116338295956371128363624655453862593 1
 1006169595007178214401008312972843537525638997252224867324240802468810244364768226 9
 8977545287852117588467627983378497928237204133439200760220176455332413095927618670 0
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(*    Conjecture 3: irrational and transcendental number ???   *)
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Conclusions

By definition, in some mode, the **Psi function** is a “part” of the **Riemann’s Zeta function**.
The zeros of the Psi function seems to be in the **critical strip** $0 < \mathrm{Re}(s) \leqslant 1$.
I proposed as **new mathematical constant** the imaginary part of the first zero of the Psi function in the semi-strip $\mathrm{Re}(s) \geqslant \frac{1}{2}$: $\mathbf{Im}(s_1) \simeq 4.9068764351$.

I’m sorry, but I have not found reference materials really regarding a function like the Psi function and the study of its zeros.

References

1) https://oeis.org/A091131
2) https://oeis.org/A058303
3) https://mathworld.wolfram.com/FactorialSums.html
4) https://mathworld.wolfram.com/RiemannZetaFunction.html